UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2011 question paper

for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/11 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- CIE will not enter into discussions or correspondence in connection with these mark schemes.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

<u>Penalties</u>

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Mark	Total
1	Any method including cover-up rule.	$\frac{1}{(2r+1)(2r+3)} = \frac{1}{2}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$	B1		
	Expresses all terms as differences.	$S_n = \frac{1}{2}\left(\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right).....\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)\right)$	M1A1		
	Finds sum.	$= \frac{1}{6} - \frac{1}{2(2n+3)}$ (acf)	A1	4	
		$S_\infty = \frac{1}{6}$ (B0M1A1√ A0A1√ if signs reversed.)	A1	1	**[5]**
2	Sum of roots.	$\frac{\beta + \beta k + \beta k^2}{k} = -p$	B1		
	Sum of products in pairs.	$\frac{\beta^2}{k} + k\beta^2 + \beta^2 = q$	B1		
	Factorises.	$\Rightarrow \beta\left(\frac{k^2+k+1}{k}\right) = -p$ and $\beta^2\left(\frac{k^2+k+1}{k}\right) = q$	M1		
		$\Rightarrow \beta = -\frac{q}{p}$ (AG)	A1	4	
	Product of roots.	$\beta^3 = -r$	B1		
		$\Rightarrow -\frac{q^3}{p^3} = -r \Rightarrow rp^3 = q^3$ (AG)	B1	2	**[6]**
3 (i)	Reduces matrix to echelon form.	$\begin{pmatrix} 1 & 3 & -2 & 4 \\ 5 & 15 & -9 & 19 \\ -2 & -6 & 3 & -7 \\ 3 & 9 & -5 & 11 \end{pmatrix}. \rightarrow \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	M1A1		
	Uses rank = dimension – nullity	r(**A**) = 4 – 2 = 2	A1	3	
(ii)	Obtains a set of equations.	$x + 3y - 2z + 4t = 0$ $z - t = 0$	B1		
	Finds basis vectors.	Basis is $\left\{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ (OE)	M1A1	3	**[6]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
4 (i)	Establishes initial result.	$f(n)+f(n+1)=3^{3n}+6^{n-1}+3^{3n+3}+6^{n}$ $=3^{3n}(1+27)+6^{n-1}(1+6)$ $=28(3^{3n})+7(6^{n-1})$ (AG)	M1 A1	2	
(ii)	States inductive hypothesis.	H_k: $f(k) = 7\lambda$	B1		
	Proves base case.	$3^3+6^0=28=4\times 7\Rightarrow$ H_1 is true	B1		
	Shows $P_k \Rightarrow P_{k+1}$.	$f(k+1)+f(k) = f(k+1)+7\lambda = 28(3^{3k})+7(6^{k-1})$ $= 7\mu$ $\Rightarrow f(k+1)=7(\mu-\lambda)$ $\therefore H_k \Rightarrow H_{k+1}$	M1		
	States conclusion.	(Hence by the principle of mathematical induction H_n is) **true for all positive integers** n.	A1	4	**[6]**
5	Right-hand loop Left-hand loop Deduct 1 mark for extra loops ($r<0$).	Position and through pole and (2, 0). Position and through pole and (2, π).	B1B1 B1B1	4	
	Uses $A=\frac{1}{2}\int r^2\,d\theta$	$A=\frac{1}{2}\int 4\cos^2\theta\,d\theta\,d\theta$	M1		
	Uses double angle formula.	$=\int(\cos 4\theta+1)\,d\theta$ (LNR)	M1		
	Integrates.	$=[\sin 4\theta+\theta]$ (LNR)	A1		
	Inserts *any appropriate* limits which legitimately give the result.	e.g. $[\sin 4\theta+\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{\pi}{2}$	A1	4	**[8]**

Qu No	Commentary	Solution	Marks	Part Mark	
6 (i)	Uses vector product to find vector perpendicular to both lines.	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 0 \\ 4 & 0 & -1 \end{vmatrix} = -3\mathbf{i} + 4\mathbf{j} - 12\mathbf{j}$	M1A1		
	Finds ***BA*** and its scalar product with unit perpendicular vector.	$\boldsymbol{BA} = \mathbf{i} + 10\mathbf{j} - 11\mathbf{j}$ perp.dist. $= \dfrac{-3\times1+4\times10+12\times11}{\sqrt{3^2+4^2+12^2}} = \dfrac{169}{\sqrt{169}} = 13$ = 13 (AG) (No penalty for sign errors made in **n**, which lead to correct result.)	M1A1	4	
(ii)		$\mathbf{p} = \begin{pmatrix} 8+4\lambda \\ 8+3\lambda \\ -7 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 7+4\mu \\ -2 \\ 4-\mu \end{pmatrix}$			
		$\mathbf{PQ} = \begin{pmatrix} -1-4\lambda+4\mu \\ -10-3\lambda \\ 11-\mu \end{pmatrix} = t\begin{pmatrix} -3 \\ 4 \\ -12 \end{pmatrix}$	B1 M1A1		
		$\Rightarrow t = -1 \quad \lambda = -2 \quad \mu = -1$	M1		
		p = 2**j** – 7**k** **q** = 3**i** – 2**j** + 5**k** (Award B1B1B1 if t assumed to be ±1 i.e. 3/5)	A1	5	[9]
6(ii)	**Alternative Solution:**				
	Finds two parameter representation for **PQ**	$\mathbf{p} = \begin{pmatrix} 8+4\lambda \\ 8+3\lambda \\ -7 \end{pmatrix}$ $\mathbf{q} = \begin{pmatrix} 7+4\mu \\ -2 \\ 4-\mu \end{pmatrix}$			
	Uses scalar product between **PQ** and direction vector of at least one line and equates to zero	$\mathbf{PQ} = \begin{pmatrix} -1-4\lambda+4\mu \\ -10-3\lambda \\ 11-\mu \end{pmatrix}$	B1		
		$16\mu - 25\lambda = 34$	M1A1		
		$17\mu - 16\lambda = 15$	A1		
	Solves simultaneously.	$\mu = -1 \qquad \lambda = -2$	M1A1		
	Obtains position vectors for P and Q	**p** = 2**j** – 7**k** **q** = 3**i** – 2**j** + 5**k**	A1	**7**	
(i)	Obtains length of PQ.	$\sqrt{3^2+(-4)^2+12^2} = 13$ (AG)	M1A1	**2**	**[9]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
7	Differentiates twice.	$v = y^3 \Rightarrow v' = 3y^2y' \Rightarrow v'' = 3y^2y'' + 6y(y')^2$	B1B1		
	Substitutes.	$\frac{1}{3}v'' - 2y(y')^2 + 2y(y')^2 + \frac{2}{3}v' - 5v = 8e^{-x}$	M1		
	Obtains v-x equation.	$\Rightarrow \frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 15v = 24e^{-x}$ (AG)	A1	4	
	Solves AQE	$m^2 + 2m - 15 = 0 \Rightarrow m = -5, 3$	M1		
	Finds CF.	CF: $Ae^{-5x} + Be^{3x}$	A1		
	Differentiates form for PI.	PI: $v = ke^{-x} \Rightarrow v' = -ke^{-x} \Rightarrow v'' = ke^{-x}$	M1		
	Substitutes.	$ke^{-x} - 2ke^{-x} - 15ke^{-x} = 24e^{-x}$	M1		
	Obtains PI	$\Rightarrow -16ke^{-x} = 24e^{-x} \Rightarrow k = -\frac{3}{2}$	A1		
	Obtains GS	GS: $v = Ae^{-5x} + Be^{3x} - \frac{3}{2}e^{-x}$	A1		
	Obtains y in terms of x (from a complete, if incorrect, GS).	$y = \left\{Ae^{-5x} + Be^{3x} - \frac{3}{2}e^{-x}\right\}^{\frac{1}{3}}$	A1√	7	**[11]**
8	Finds characteristic equation and solves.	Det(**A** – λ**I**) = 0 $\Rightarrow \lambda^3 - 4\lambda^2 - 11\lambda + 30 = 0$ $\Rightarrow \lambda = -3, 2, 5$ (Any one) (All three)	M1A1 A1 A1		
	Uses vector product (or equations) to find corresponding eigenvectors.	$\lambda = -3 \quad \mathbf{e}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -1 & 1 \\ -1 & 3 & -3 \end{vmatrix} = \begin{pmatrix} 0 \\ 20 \\ 20 \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	M1A1		
		$\lambda = 2 \quad \mathbf{e}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -1 & -2 & -3 \end{vmatrix} = \begin{pmatrix} 5 \\ 5 \\ -5 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$	A1		
		$\lambda = 5 \quad \mathbf{e}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ -1 & -5 & -3 \end{vmatrix} = \begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix} = t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$	A1	8	
	Forms **P** from eigenvectors and **D** with fifth powers of eigenvalues on leading diagonal. Note: Columns of **P** and **D** can be permuted, but must match.	$\mathbf{P} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} -243 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 3125 \end{pmatrix}$	B1√ M1 A1√	3	**[11]**

Qu No	Commentary	Solution	Marks	Part Mark	Total
9	Uses formulae for coords of centroid.	$\bar{x} = \dfrac{\int_1^4 x^{\frac{5}{2}}\mathrm{d}x}{\int_1^4 x^{\frac{3}{2}}\mathrm{d}x}$ $\quad \bar{y} = \dfrac{\frac{1}{2}\int_1^4 x^3\mathrm{d}x}{\int_1^4 x^{\frac{3}{2}}\mathrm{d}x}$	M1M1		
	Numerators.	$\left[\frac{2}{7}x^{\frac{7}{2}}\right]_1^4 = \frac{254}{7}$ $\quad \left[\frac{1}{8}x^4\right]_1^4 = \frac{255}{8}$	A1A1		
	Denominator.	$\left[\frac{2}{5}x^{\frac{5}{2}}\right]_1^4 = \frac{62}{5}$	B1		
	Obtains values. (Accept rational values.)	$\bar{x} = \frac{635}{217}$ (= 2.93) $\quad \bar{y} = \frac{1275}{496}$ (= 2.57)	M1A1	7	
	Differentiates y wrt x and squares.	$y = x^{\frac{3}{2}} \Rightarrow y' = \frac{3}{2}\sqrt{x} \Rightarrow (y')^2 = \frac{9x}{4}$	B1		
	Subs in arc length formula (LR).	$s = \int_5^{28}\sqrt{1+\frac{9x}{4}}\,\mathrm{d}x$	B1		
	Integrates.	$= \left[\frac{8}{27}\left(1+\frac{9x}{4}\right)^{\frac{3}{2}}\right]_5^{28}$ or $\left[\frac{1}{27}(4+9x)^{\frac{3}{2}}\right]_5^{28}$	M1A1		
	Obtains printed result.	$= \frac{8^4-7^3}{27}$ or $\frac{16^3-7^3}{27} = 139$ (AG)	A1	5	[12]

Qu No	Commentary	Solution	Marks	Part Mark	Total
10	Puts $\cos^n x = \cos^{n-1} x.\cos x$ and integrates by parts.	$I_n = \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$	M1A1		
	Uses $\cos^2 x = 1 - \sin^2 x$ and obtains reduction formula.	$I_n = (n-1)\int_0^{\frac{\pi}{2}} (\cos^{n-2} x - \cos^n x) \, dx$	M1		
		$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$ ($n \geq 2$) (AG)	A1	4	
	Uses mean value formula.	Mean value = $\frac{\int_a^b y dx}{b-a}$	M1		
	Transforms variable to t.	$= \frac{\int_0^{\frac{\pi}{2}} a\cos^3 t.3a\sin^2 t\cos t dt}{a}$	M1A1		
	Uses $\sin^2 t = 1 - \cos^2 t$ to write in a form where reduction formula can be used.	$= 3a\int_0^{\frac{\pi}{2}} \cos^4 t \sin^2 t dt$	A1	4	
		$= 3a\int_0^{\frac{\pi}{2}} (\cos^4 t - \cos^6 t) dt$ (AG)			
	Finds I_0. Uses reduction formula to find I_4 and I_6.	$I_0 = \frac{\pi}{2}$ or $I_2 = \pi/4$.	B1		
		$I_4 = \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3}{16}\pi$, $I_6 = \frac{5}{6}.\frac{3}{16}.\pi = \frac{5}{32}\pi$	M1A1		
	(Or equivalent.)	(e.g. $I_6 = \frac{5}{6} I_4 \Rightarrow (I_4 - I_6) = \frac{1}{6} I_4$)			
	Substitutes in integral to find mean value.	Mean value = $\frac{3a}{32}\pi$	A1	4	[12]

Qu No	Commentary	Solution	Marks	Part Mark	
11	**EITHER**				
	Uses de Moivre's theorem and binomial theorem.	$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ and $\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$	B1 M1A1		
	(If line 1 missing, or no reference made to cos3θ and sin3θ being real and imaginary parts.)	Award B0M1A1M1M1A0 i.e.4/6			
	Equates real and imaginary parts.	$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$	M1		
	Uses $\tan A = \dfrac{\sin A}{\cos A}$	$\tan 3\theta = \dfrac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$	M1		
		$\therefore \tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ (*) (AG)	A1	6	
		$\dfrac{\pi}{12}$, $\dfrac{5\pi}{12}$, $\dfrac{9\pi}{12} = \dfrac{3\pi}{4}$	B1 (one) B1 (all)	2	
		Put $\tan 3\theta = 1$ in (*) $\Rightarrow t^3 - 3t^2 - 3t + 1 = 0$	M1		
	States roots of $\tan 3\theta = 1$ between 0 and π.	Roots are $\tan\dfrac{\pi}{12}, \tan\dfrac{5\pi}{12}, \tan\dfrac{3\pi}{4}$	A1 (one) A1 (all)	3	
	Obtains cubic equation and solves.	$(t+1)(t^2 - 4t + 1) = 0$ $\Rightarrow t = -1$, $2 \pm \sqrt{3}$	M1		
	Evaluates each root.	$\tan\dfrac{3\pi}{4} = -1$ $\tan\dfrac{\pi}{12} = 2 - \sqrt{3}$ $\tan\dfrac{5\pi}{12} = 2 + \sqrt{3}$	A1 (one) A1 (all)	3	[14]

Qu No	Commentary	Solution	Marks	Part Mark	Total
11	**OR**				
(i)	Differentiates y wrt x.	$\frac{dy}{dx}=\frac{(x+3)(2x+\lambda)-(x^2+\lambda x-6\lambda^2)}{(x+3)^2}$	M1		
		$=\frac{x^2+6x+3\lambda+6\lambda^2}{(x+3)^2}$	A1		
		$\frac{dy}{dx}=0$ has distinct roots if			
	Uses discriminant and factorises.	$36-12\lambda-24\lambda^2>0 \Rightarrow 3-\lambda-2\lambda^2>0$ $(3+2\lambda)(1-\lambda)>0$	M1 A1		
	Obtains result. (No equality.)	$\Rightarrow -\frac{3}{2}<\lambda<1$ (AG)	A1	5	
(ii)	Uses division to obtain the form for recognition of oblique asymptote.	$\frac{x^2+\lambda x-6\lambda^2}{x+3}\equiv x+\lambda-3+\frac{9-3\lambda-6\lambda^2}{x+3}$ (Tolerate error on remainder term.)	M1		
		$\Rightarrow$ asymptotes are: $x=-3$ and $y=x+\lambda-3$.	B1 A1	3	
(iii)	For $0<\lambda<1$	Axes and asymptotes. Upper branch with minimum below x-axis. Lower branch with maximum.	B1 B1 B1	3	
(iv)	For $\lambda>3$	Axes and asymptotes. (n.b. oblique asymptote has positive y intercept.) Left-hand branch. Right-hand branch.	B1 B1 B1	3	
	Deduct 1 mark overall for wrong forms At infinity.				**[14]**